AN ITERATIVE PROCESS IN THE PROBLEM OF PLATEAU*

BY

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Introduction

The problem of Plateau calls for a minimal surface bounded by a given curve. In analytic formulation, the problem requires the determination of three functions $x(u, v)$, $y(u, v)$, $z(u, v)$, subject to certain boundary conditions and such that

$$\text{(I)} \qquad x(u, v),\ \ y(u, v),\ \ z(u, v) \text{ are harmonic,}$$

and

$$\text{(II)} \qquad E = G,\ \ F = 0,$$

where

$$E = x_u^2 + y_u^2 + z_u^2, F = x_u x_v + y_u y_v + z_u z_v, G = x_v^2 + y_v^2 + z_v^2.$$

In my previous work† on this problem, I introduced the notion of *approximate solutions* of the problem by replacing the above *exact condition* (II) by the *approximate condition* that the two integrals

$$\iint (E^{1/2} - G^{1/2})^2 dudv \quad\text{and}\quad \iint |F|\, dudv$$

be small. The boundary conditions were also replaced by approximate conditions. I gave a direct construction for the approximate solutions; the exact solution was then obtained by a passage to the limit.

The main purpose of the present paper is to develop an *iterative process* which produces automatically a sequence of approximate solutions. The idea of the process (described and discussed in §2) is to comply with the above conditions (I) and (II) alternately. The process starts with an arbitrary harmonic surface bounded by the given curve and thus the result depends, in general, upon the choice of this initial harmonic surface. In this sense, the process contains an *arbitrary parameter*, namely the initial harmonic surface $\mathfrak{H}_0$, and this fact accounts for the great flexibility of the method.

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† *On Plateau's problem*, Annals of Mathematics, (2), vol. 31 (1930), pp. 457–469; *The problem of the least area and the problem of Plateau*, Mathematische Zeitschrift, vol. 32 (1930), pp. 763–796.

To illustrate this point, let us recall that in general the solution of the problem of Plateau is not unique.† The construction of the approximate solutions, used in my previous work, yielded a solution of the problem whose area was a minimum, and which consequently was not the general solution of the problem, since a minimal surface, in general, does not have a minimum area.‡ On the other hand, it is rather obvious that the iterative process yields *all the solutions* of the problem, including those with minimum area, if the parameter $\mathfrak{H}_0$ is chosen in all possible ways. For the sake of simplicity, this very trivial remark will be verified only for the case when the given boundary curve is a polygon. On the other hand, we shall see that the iterative process, if properly applied, can also be made to yield a *solution with a minimum area.*

After the approximate solutions have been constructed, a passage to the limit is necessary to obtain an exact solution. In my previous work, this passage to the limit has been carried out under the assumption that the given boundary curve bounds at least one continuous surface with a finite area. Replacing a lemma of Courant which I used in my proof by a lemma due to J. Douglas, it follows however that the restriction mentioned above can be dropped. Thus it follows that in order to obtain the solution for a general Jordan curve, it is sufficient to secure approximate solutions (which can be constructed directly); it is not necessary to solve the problem of Plateau first for some special class of curves.

Summing up, we have the following picture. Given a general Jordan curve, we can always construct approximate solutions of the problem of Plateau, and a passage to the limit yields then an exact solution. In case the given curve has a finite minimum area, the approximate solutions can be constructed in such a way as to obtain an exact solution with a minimum area. The construction of the approximate solutions can be based on an automatic iterative process, properly applied. In case the given curve is a polygon, the iterative process can be shown to yield all the solutions of the problem.

We conclude this introduction with a few remarks concerning the method used in this paper. The characteristic feature of the method is the essential use of *conformal mapping*. Indeed, the method is based on the following operation. Given a surface

$$S: x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad u^2 + v^2 \leqq 1;$$

first, change to isothermic parameters, that is to say, change to a representation

$$S: x = x^*(u, v), y = y^*(u, v), z = z^*(u, v), u^2 + v^2 \leqq 1,$$

† See also for literature the author's paper *Contributions to the theory of minimal surfaces*, Acta Szeged, vol. 6 (1932), pp. 1–20.

‡ See Radó, Acta Szeged, loc. cit.

such that $E^*=G^*$, $F^*=0$. Secondly, take the harmonic functions $\bar{x}(u, v)$, $\bar{y}(u, v)$, $\bar{z}(u, v)$ which coincide with $x^*(u, v)$, $y^*(u, v)$, $z^*(u, v)$ on $u^2+v^2=1$. Thus we derive from S a harmonic surface

$$\mathfrak{H}: x = \bar{x}(u, v),\ y = \bar{y}(u, v),\ z = \bar{z}(u, v),\ u^2 + v^2 \leqq 1.$$

This operation, leading from S to $\mathfrak{H}$, and the simple relations between S and $\mathfrak{H}$, were the essential tools which I used to construct the approximate solutions in my previous work†, and these same tools will be used also in the present paper in setting up the iterative process. In order to get from S to $\mathfrak{H}$, we have to use a conformal map of S, and to get around the difficulties arising in this connection, in my previous work I approximated S by polyhedrons and referred to the theorem, proved by H. A. Schwarz, that polyhedrons do admit of conformal maps (in a properly generalized sense). Since my first publications on this subject, very substantial progress has been achieved in the theory of the conformal mapping of general surfaces.‡ A theorem of McShane, concerned with conformal maps of saddle-shaped surfaces, permits us to deal directly with the surfaces which arise in the course of the iterative process and to present this process in a very compact and elegant way.

1. Preliminaries

1.1. In this section, the definitions, lemmas and theorems referred to in the sequel will be stated for the convenience of the reader.

A *Jordan arc*, in the xyz-space, is a one-to-one and continuous image of an interval $a \leqq t \leqq b$. A *Jordan curve* is the one-to-one and continuous image of the unit circle $u=\cos\theta$, $v=\sin\theta$.

If C_1 and C_2 are two Jordan arcs, then their *distance* $d(C_1, C_2)$, in the Fréchet sense, is defined as follows. Denote by τ a one-to-one and continuous correspondence between C_1 and C_2, and let $M(\tau)$ denote the maximum distance of corresponding points. The distance $d(C_1, C_2)$ is the greatest lower bound of $M(\tau)$, for all possible choices of τ. The distance of two Jordan curves is defined in the same manner. *Convergent sequences* are then defined in terms of the distance.

1.2. Given two Jordan arcs C_1 and C_2, suppose there is given a transfor-

† See second foot note on p. 869. This passage from S to $\mathfrak{H}$, and the simple inequalities concerning the relations between S and $\mathfrak{H}$, have been subsequently used by J. Douglas and E. J. McShane also. See J. Douglas, *Solution of the problem of Plateau*, these Transactions, vol. 33 (1931), pp. 263–321, and *The mapping theorem of Koebe and the problem of Plateau*, Journal of Mathematics and Physics of the Massachusetts Institute of Technology, vol. 10 (1931), pp. 106–130; E. J. McShane, these Transactions, vol. 35 (1933), pp. 716–733.

‡ McShane, loc. cit. Further important contributions to the theory of the conformal maps of general surfaces are contained in several as yet unpublished papers by C. B. Morrey, which the author has had the privilege to see in manuscript.

mation T which associates with every point of C_1 a definite point of C_2 (the existence of an inverse transformation is not required). T will be called a *monotonic transformation* of C_1 into a set on C_2 if the following conditions are satisfied. Whenever distinct points P_1, Q_1, R_1 on C_1 are such that Q_1 is on the sub-arc with end points P_1, R_1, then their images P_2, Q_2, R_2 on C_2 have the same relative positions; in case P_2 and R_2 coincide, it is required that Q_2 also coincide with them.

For two Jordan curves Γ_1^*, Γ_2^*, a monotonic transformation T† of Γ_1^* into a set on Γ_2^* is then defined as follows. There exists a triple of points A_1^*, B_1^*, C_1^* on Γ_1^* with distinct images A_2^*, B_2^*, C_2^*, such that the three non-overlapping arcs $A_1^*B_1^*$, $B_1^*C_1^*$, $C_1^*A_1^*$ of Γ_1^* are taken by T in a monotonic way into sets on the three non-overlapping arcs $A_2^*B_2^*$, $B_2^*C_2^*$, $C_2^*A_2^*$ of Γ_2^*. The notion of a continuous monotonic transformation is then self-explanatory.

1.3. The term *monotonic transformation* has been chosen to suggest that such transformations have a number of properties in common with monotonic functions. As a consequence, a simple and important selection theorem of Helly, concerned with sequences of monotonic functions, generalizes immediately to sequences of monotonic transformations in the following manner.‡

Let there be given a Jordan curve γ, and a sequence of Jordan curves Γ_n^* converging toward a Jordan curve Γ^* in the Fréchet sense. Given three distinct points a, b, c on γ, three distinct points A^*, B^*, C^* on Γ^*, and three distinct points A_n^*, B_n^*, C_n^* on Γ_n^*, such that $A_n^* \to A^*$, $B_n^* \to B^*$, $C_n^* \to C^*$. Consider any sequence of monotonic transformations T_n, such that T_n carries γ into a set on Γ_n^* and a, b, c into A_n^*, B_n^*, C_n^*. Then there exists a subsequence T_{n_k}, such that for every point p of γ the sequence of the points $P_{n_k}^*$, which correspond to p under T_{n_k}, converge toward a definite point P^* on Γ^*. The transformation which associates with p this limit point P^* is a monotonic transformation T of γ into a set on Γ^*. The limit transformation T carries a, b, c into A^*, B^*, C^*.

1.4. If a sequence of monotonic functions converges, in a closed interval, toward a continuous function, then the convergence necessarily is uniform.§ As observed by McShane in conversation with the author, this holds ob-

† J. Douglas, in his work on the problem of Plateau, speaks of proper and improper parametric representations, and obtains the necessary facts by an interpretation on the torus. The term monotonic transformation, used in my own work, calls attention to the analogy with monotonic functions; the necessary facts appear then as immediate consequences of this analogy.

‡ See references cited in second footnote on p. 869.

§ See H. E. Buchanan and T. H. Hildebrandt, *Note on the convergence of a sequence of functions of a certain type*, Annals of Mathematics, vol. 9 (1908), p. 123.

viously for sequences of monotonic transformations also. Thus we have the following corollary to the preceding selection theorem: if the limit transformation T is continuous, then T_{n_k} converges *uniformly* toward T, the meaning of this assertion being too obvious to be explained.

1.5. Suppose now we have a monotonic transformation T of the unit circle $u=\cos\theta$, $v=\sin\theta$ into a set on a Jordan curve Γ^*. Then T can be given by a set of equations

$$T\colon x = \xi(\theta),\ y = \eta(\theta),\ z = \zeta(\theta),$$

and, in analogy with monotonic functions, we have the following statements.

(a) If T is not a one-to-one transformation, then there exists an arc σ on $u^2+v^2=1$, such that $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ all three reduce to constants on σ.

(b) For every θ_0, $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ have definite one-sided limits ξ_0^+, ξ_0^-, η_0^+, η_0^-, ζ_0^+, ζ_0^-.

(c) If, for a certain θ_0, we have $\xi_0^+=\xi_0^-$, $\eta_0^+=\eta_0^-$, $\zeta_0^+=\zeta_0^-$, then $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ are continuous at θ_0.

(d) The points of discontinuity of $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ form a denumerable set.

1.6. A *continuous surface* S, *of the topological type of the circular disc*, is defined by a set of equations

$$S\colon x = x(u, v),\ y = y(u, v),\ z = z(u, v),\ (u, v) \text{ in } R, \tag{1}$$

where R is some Jordan region (that is, the set of points in and on a Jordan curve in the uv-plane), and $x(u, v)$, $y(u, v)$, $z(u, v)$ are continuous in R. Given then another such surface

$$\bar{S}\colon x = \bar{x}(u, v),\ y = \bar{y}(u, v),\ z = \bar{z}(u, v),\ (u, v) \text{ in } \bar{R}, \tag{2}$$

the *distance* $d(S, \bar{S})$, in the Fréchet sense, is defined as follows. Let τ denote a one-to-one and continuous correspondence between R and $\bar{R}$, and denote by (u, v), $(\bar{u}, \bar{v})$ a couple of points corresponding under τ. Denote by P the point of S corresponding to (u, v), by $\bar{P}$ the point of $\bar{S}$ corresponding to $(\bar{u}, \bar{v})$, and by $M(\tau)$ the maximum distance of P and $\bar{P}$, for all choices of the couple (u, v), $(\bar{u}, \bar{v})$. Then $d(S, \bar{S})$ is the greatest lower bound of $M(\tau)$, for all possible choices of τ.

If $d(S, \bar{S})=0$, then $\bar{S}$ is considered as identical to S, and (1) and (2) are considered as parametric representations of the same surface.

Convergent sequences of surfaces are defined in terms of the distance in an obvious manner.

A continuous surface S is bounded by a Jordan curve Γ^* if it admits of a representation (1) such that the boundary curve of R is taken in a topological way into Γ^*.

1.7. A continuous surface will be called a *polyhedron* and will be denoted by $\mathfrak{P}$, if it admits of a parametric representation

$$(3) \qquad \mathfrak{P}: x = x(u, v),\ y = y(u, v),\ z = z(u, v),\ (u, v) \text{ in } R,$$

with the following properties. R can be subdivided into a finite number of curvilinear triangles $\delta_1, \cdots, \delta_m$, every one of which is carried by (3) in a topological way into a non-degenerate plane rectilinear triangle in the xyz-space. These rectilinear triangles will be denoted by $\Delta_1, \cdots, \Delta_m$. It is furthermore required that the boundary curve of R be carried by (3) in a topological way into a simple closed polygon $\mathfrak{p}^*$, which will be called the boundary polygon of $\mathfrak{P}$. A representation (3) with these properties will be called a *typical representation* of $\mathfrak{P}$.

1.8. A fundamental theorem, proved already by H. A. Schwarz, asserts the existence of conformal maps of polyhedrons, in the following sense. Given a polyhedron $\mathfrak{P}$, there exists a representation which is typical in the sense of §1.7 and possesses the following additional properties.†

(a) The region R is the unit circle $u^2+v^2 \leqq 1$.

(b) The sides of the curvilinear triangles $\delta_1, \cdots, \delta_m$ are analytic arcs including their end points; none of these triangles has a zero angle.

(c) $x(u, v), y(u, v), z(u, v)$ are analytic in the interior of $\delta_1, \cdots, \delta_m$, and satisfy there the relations $E=G$, $F=0$.

(d) Three points A, B, C, given arbitrarily on $u^2+v^2=1$, are carried into three points A^*, B^*, C^* arbitrarily given on the boundary polygon $\mathfrak{p}^*$ of $\mathfrak{P}$.

1.9. The sum of the areas of the triangles $\Delta_1, \cdots, \Delta_m$, defined in §1.7, is the area $\mathfrak{A}(\mathfrak{P})$ of $\mathfrak{P}$. The *area* $\mathfrak{A}(S)$, in the Lebesgue sense, of a continuous surface S is then defined as follows. Consider a sequence of polyhedrons $\mathfrak{P}_n$ converging toward S in the Fréchet sense. $\mathfrak{A}(S)$ is then the greatest lower bound of lim inf $\mathfrak{A}(\mathfrak{P}_n)$ for all choices of the sequence $\mathfrak{P}_n$.‡

1.10. From the existence of conformal maps of polyhedrons McShane§ derived an important existence theorem for saddle-surfaces. To state this theorem, the following definitions are necessary.

A function $f(u, v)$, defined in $u^2+v^2 \leqq 1$, satisfies condition (C) if the following hold:

(a) $f(u, v)$ is continuous in $u^2+v^2 \leqq 1$.

† The reader may consult the beautiful book by Carathéodory, *Conformal Representation* (Cambridge University Press), Chapter VII.

‡ For a systematic presentation of the theory of the area and for literature, see the author's paper *Über das Flächenmass rektifizierbarer Flächen*, Mathematische Annalen, vol. 100 (1928), pp. 445–479. Quite recently, important contributions have been made to the theory by McShane (see for references McShane, loc. cit.) and by C. B. Morrey (in several as yet unpublished papers).

§ McShane, these Transactions, loc. cit.

(b) $f(u, v)$ is, for almost every value of u, an absolutely continuous function of v, and for almost every value of v an absolutely continuous function of u.

(c) The Dirichlet integral $\iint(f_u^2+f_v^2)$, taken over $u^2+v^2<1$, is finite.

1.11. A function $f(u, v)$, continuous in a Jordan region R, is *monotonic* there if for every domain D (connected open set) in R it is true that $m_b \leqq f(u, v) \leqq M_b$ in D, where m_b, M_b denote the minimum and maximum respectively of $f(u, v)$ on the boundary of D.

1.12. A continuous surface (1) is called a *saddle-surface* if $x(u, v)$, $y(u, v)$, $z(u, v)$ are monotonic. This property is independent of the parametric representation.†

1.13. The *theorem of McShane* reads then as follows. Given a continuous surface

$$S: x = x(u, v),\ y = y(u, v),\ z = z(u, v),\ u^2 + v^2 \leqq 1, \tag{4}$$

suppose that

(a) S is a saddle-surface;

(b) $\mathfrak{A}(S)$ is finite;

(c) the equations (4) define a continuous monotonic transformation of $u^2+v^2=1$ into a Jordan curve Γ^*.

Then there exists a representation of S,

$$S: x = \bar{x}(u, v),\ y = \bar{y}(u, v),\ z = \bar{z}(u, v),\ u^2 + v^2 \leqq 1, \tag{5}$$

which has the following properties.

(α) $\bar{x}(u, v)$, $\bar{y}(u, v)$, $\bar{z}(u, v)$ satisfy condition (C) of §1.10.

(β) $\overline{E}=\overline{G}$, $\overline{F}=0$ almost everywhere in $u^2+v^2<1$.

(γ) The equations (5) define again a continuous monotonic transformation of $u^2+v^2=1$ into Γ^*.

(δ) Three points A, B, C, given arbitrarily on $u^2+v^2=1$, are carried by (5) into three points A^*, B^*, C^* given arbitrarily on Γ^*.

(ϵ) $\mathfrak{A}(S)$ is given by the usual integral formula, which reduces, on account of (β), to

$$\mathfrak{A}(S) = \tfrac{1}{2}\iint_{u^2+v^2<1}(\overline{E} + \overline{G})dudv.$$

1.14. Given a Jordan curve Γ^* in the xyz-space, the *problem of Plateau* for Γ^* will be stated as follows. Determine three functions $x(u, v)$, $y(u, v)$, $z(u, v)$ with the following properties.

(a) $x(u, v)$, $y(u, v)$, $z(u, v)$ are continuous in $u^2+v^2\leqq 1$, harmonic in $u^2+v^2<1$, and

† McShane, these Transactions, loc. cit.

(b) satisfy in $u^2+v^2<1$ the equations $E=G$, $F=0$.

(c) The equations $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$ carry $u^2+v^2=1$ in a topological way into Γ^*.

1.15. Given, in a domain D, three functions $x(u, v)$, $y(u, v)$, $z(u, v)$, we shall say that they form a *triple of conjugate harmonic functions*, if they are harmonic and satisfy the equations $E=G$, $F=0$ in D. If one of the three functions, say $z(u, v)$, vanishes identically, then $x(u, v)$ and $y(u, v)$ are obviously conjugate harmonic functions in the sense used in theory of functions. This analogy between minimal surfaces on the one hand and analytic functions of a complex variable $w=u+iv$ on the other hand has always been of fundamental importance in the theory of minimal surfaces.

We shall need the following two lemmas.†

1.16. Suppose we have, in $u^2+v^2<1$, a triple of conjugate harmonic functions $x(u, v)$, $y(u, v)$, $z(u, v)$ which remain continuous on $u^2+v^2=1$, and all three reduce to constants on a certain arc of $u^2+v^2=1$. Then $x(u, v)$, $y(u, v)$, $z(u, v)$ reduce to constants identically.

1.17. Let there be given, in a sector $0<u^2+v^2<r^2$, $0<\arctan (v/u)<\alpha$, a triple of conjugate harmonic functions $x(u, v)$, $y(u, v)$, $z(u, v)$. Suppose that these functions remain continuous on $v=0$, $0<u<r$, and that $x(u, 0)$, $y(u, 0)$, $z(u, 0)$ approach definite finite limits x_0, y_0, z_0 for $u\to+0$. Then $x(u, v)\to x_0$, $y(u, v)\to y_0$, $z(u, v)\to z_0$ if $(u, v)\to(0, 0)$ in any subsector

$$0 < u^2 + v^2 < r^2, \quad 0 \leqq \arctan \frac{v}{u} \leqq \beta < \alpha.$$

1.18. Let there be given, on the unit circle $u=\cos\theta$, $v=\sin\theta$, three functions $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$. Suppose that $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ are summable and that at a certain θ_0 they have definite finite one-sided limits ξ_0^+, ξ_0^-, η_0^+, η_0^-, ζ_0^+, ζ_0^-. Suppose that the harmonic functions $x(u, v)$, $y(u, v)$, $z(u, v)$, obtained by means of the Poisson integral formula by using $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ as boundary functions, constitute a triple of conjugate harmonic functions in $u^2+v^2<1$. Then

$$\xi_0^+ = \xi_0^-, \ \eta_0^+ = \eta_0^-, \ \zeta_0^+ = \zeta_0^-.$$

This lemma, due to J. Douglas‡, can be obtained as an immediate consequence of the generalized Lindelöf theorem, stated in §1.16.§

1.19. Given, in the xyz-space, a Jordan curve Γ^*, we shall consider in the

† For proofs and literature concerning the lemmas in 1.16 and 1.17, which generalize classical theorems of Schwarz and of Lindelöf, see E. F. Beckenbach and T. Radó, *Subharmonic functions and minimal surfaces*, these Transactions, vol. 35 (1933), p. 648–661.

‡ Loc. cit. in first footnote on p. 871.

§ Beckenbach and Radó, loc. cit.

sequel an approximate form of the problem of Plateau so often that it is convenient to have a symbol for it. Given three distinct points A, B, C on $u^2+v^2=1$, three distinct points A^*, B^*, C^* on Γ^*, and an $\epsilon>0$. We shall then denote by $P(\Gamma^*; A, B, C, A^*, B^*, C^*; \epsilon)$ the following problem.† Determine three functions $x(u, v)$, $y(u, v)$, $z(u, v)$ with the following properties.

(a) $x(u, v)$, $y(u, v)$, $z(u, v)$ are continuous in $u^2+v^2\leqq 1$, harmonic in $u^2+v^2<1$, and

(b) satisfy the relations

$$\iint (E^{1/2} - G^{1/2})^2 \leqq \epsilon, \quad \iint |F| \leqq \epsilon,$$

the integrals being taken over $u^2+v^2<1$.

(c) The equations

$$x = x(u, v),\ y = y(u, v),\ z = z(u, v),\ u^2 + v^2 = 1, \tag{6}$$

define a continuous monotonic transformation (see §1.2) of $u^2+v^2=1$ into a (not prescribed) Jordan curve $\bar{\Gamma}^*$, such that the distance of Γ^* and $\bar{\Gamma}^*$ is $\leqq\epsilon$. Finally, A, B, C are taken by (6) into three distinct points $\bar{A}^*$, $\bar{B}^*$, $\bar{C}^*$ on $\bar{\Gamma}^*$, such that the distances $A^*\bar{A}^*$, $B^*\bar{B}^*$, $C^*\bar{C}^*$ are all three $\leqq\epsilon$.

1.20. It is important to observe that the problem $P(\Gamma^*; A, B, C, A^*, B^*, C^*; 0)$ is identical to the problem of Plateau, as stated in §1.14. Indeed, if $\epsilon=0$, then condition (b) in §1.19 reduces to $E=G$, $F=0$ in $u^2+v^2<1$, since E, F, G are continuous (even analytic). We have to see what happens on the boundary. If $\epsilon=0$, then condition (c) in §1.19 only requires that the equations (6) define a continuous and monotonic transformation T of $u^2+v^2=1$ into Γ^*. We must show that T is a one-to-one transformation. However, since $E=G$, $F=0$ has already been verified, the one-to-one character of T follows directly from §1.16.

2. The iterative process

2.1. Let there be given, in the xyz-space, a Jordan curve Γ^*. Take three distinct points A, B, C on $u^2+v^2=1$, and three distinct points A^*, B^*, C^* on Γ^*. Suppose there is given a triple of functions $x_0(u, v)$, $y_0(u, v)$, $z_0(u, v)$ with the following properties.

(α) $x_0(u, v)$, $y_0(u, v)$, $z_0(u, v)$ are continuous in $u^2+v^2\leqq 1$ and harmonic in $u^2+v^2<1$.

(β) The equations

$$x = x_0(u, v),\ y = y_0(u, v),\ z = z_0(u, v),\ u^2 + v^2 = 1,$$

† The approximate form of the problem of Plateau has been introduced in the author's papers in the Annals of Mathematics and Mathematische Zeitschrift, loc. cit.

define a continuous monotonic transformation of $u^2+v^2=1$ into Γ^*, such that A, B, C are taken into A^*, B^*, C^*.

(γ) The area of the surface

$$\mathfrak{H}_0\colon x = x_0(u, v),\ y = y_0(u, v),\ z = z_0(u, v),\ u^2 + v^2 \leqq 1, \tag{7}$$

is finite.

On account of condition (α), the area of $\mathfrak{H}_0$ is then given by

$$\mathfrak{A}(\mathfrak{H}_0) = \iint (E_0G_0 - F_0^2)^{1/2}$$

where E_0, F_0, G_0 are the first fundamental quantities relative to the representation (7).

2.2. The preceding assumptions being satisfied, the iterative process runs as follows. On account of conditions (α), (β), (γ) the theorem of McShane (§1.13) applies to $\mathfrak{H}_0$, and we have therefore a representation

$$\mathfrak{H}_0\colon x = \bar{x}_0(u, v),\ y = \bar{y}_0(u, v),\ z = \bar{z}_0(u, v),\ u^2 + v^2 \leqq 1 \tag{8}$$

which satisfies the following conditions.

($\bar{\alpha}$) $\bar{x}_0(u, v)$, $\bar{y}_0(u, v)$, $\bar{z}_0(u, v)$ satisfy condition (C) of §1.10 in $u^2+v^2\leqq 1$, and also satisfy there the relations $\overline{E}_0=\overline{G}_0$, $\overline{F}_0=0$ almost everywhere ($\overline{E}_0$, $\overline{F}_0$, $\overline{G}_0$ are the first fundamental quantities relative to the representation (8)).

($\bar{\beta}$) The equations (8) define, for $u^2+v^2=1$, a continuous monotonic transformation of $u^2+v^2=1$ into Γ^*, such that A, B, C are carried into A^*, B^*, C^*.

($\bar{\gamma}$) On account of ($\bar{\alpha}$), $\mathfrak{A}(\mathfrak{H}_0)$ is given by

$$\mathfrak{A}(\mathfrak{H}_0) = \tfrac{1}{2}\iint (\overline{E}_0 + \overline{G}_0). \tag{9}$$

Denote then by $x_1(u, v)$, $y_1(u, v)$, $z_1(u, v)$ the harmonic functions coinciding with $\bar{x}_0(u, v)$, $\bar{y}_0(u, v)$, $\bar{z}_0(u, v)$ on $u^2+v^2=1$, and define the surface $\mathfrak{H}_1$ by

$$\mathfrak{H}_1\colon x = x_1(u, v),\ y = y_1(u, v),\ z = z_1(u, v),\ u^2 + v^2 \leqq 1.$$

Let us first verify that $\mathfrak{H}_1$ has also a finite area. Since a harmonic function with given boundary values minimizes the Dirichlet integral† we have the inequalities

$$\iint (x_{1u}^2 + x_{1v}^2) \leqq \iint (\bar{x}_{0u}^2 + \bar{x}_{0v}^2), \quad \iint (y_{1u}^2 + y_{1v}^2) \leqq \iint (\bar{y}_{0u}^2 + \bar{y}_{0v}^2),$$

$$\iint (z_{1u}^2 + z_{1v}^2) \leqq \iint (\bar{z}_{0u}^2 + \bar{z}_{0v}^2),$$

† See Hurwitz-Courant, *Funktionentheorie* (Berlin, Springer, 1922), p. 335, Hilfsatz II. The proof, as given there, covers the present case on account of condition (C) stated in §1.10 (see McShane, loc. cit.).

and hence, by addition,

$$\iint (E_1 + G_1) \leqq \iint (\overline{E}_0 + \overline{G}_0), \tag{10}$$

the integrals being taken over $u^2+v^2<1$. Since

$$(E_1G_1 - F_1^2)^{1/2} \leqq E_1^{1/2}G_1^{1/2} \leqq \tfrac{1}{2}(E_1 + G_1),$$

it follows from (9) and from (10) that the integral of $(E_1G_1-F_1^2)^{1/2}$ taken over $u^2+v^2<1$, that is to say, the area of $\mathfrak{H}_1$, is finite. For further use we note the obvious inequalities

$$\begin{aligned} \mathfrak{A}(\mathfrak{H}_1) &= \iint (E_1G_1 - F_1^2)^{1/2} \leqq \iint E_1^{1/2}G_1^{1/2} \leqq \frac{1}{2}\iint (E_1 + G_1) \\ &\leqq \frac{1}{2}\iint (\overline{E}_0 + \overline{G}_0) = \mathfrak{A}(\mathfrak{H}_0) = \iint (E_0G_0 - F_0^2)^{1/2} \\ &\leqq \iint E_0^{1/2}G_0^{1/2} \leqq \frac{1}{2}\iint (E_0 + G_0). \end{aligned} \tag{11}$$

Since the area of $\mathfrak{H}_1$ is finite, we can repeat the procedure by which we derived $\mathfrak{H}_1$ from $\mathfrak{H}_0$. We obtain in this way a surface $\mathfrak{H}_2$, to which we can apply the same procedure, and so on indefinitely. We obtain in this way a sequence of surfaces

$$\mathfrak{H}_n\colon x = x_n(u, v),\ y = y_n(u, v),\ z = z_n(u, v),\ u^2 + v^2 \leqq 1, \tag{12}$$

where $x_n(u, v)$, $y_n(u, v)$, $z_n(u, v)$ satisfy the conditions (α), (β), (γ) listed in §2.1 (the points A, B, C, A^*, B^*, C^* are kept fixed).

2.3. We put

$$\epsilon_n = \max\left(\iint (E_n^{1/2} - G_n^{1/2})^2,\ \iint |F_n|\right).\dagger$$

We assert that $\epsilon_n \to 0$. In other words, $x_n(u, v)$, $y_n(u, v)$, $z_n(u, v)$ constitute a solution of the problem $P(\Gamma^*; A,B,C,A^*, B^*, C^*; \epsilon_n)$, where $\epsilon_n \to 0$. Furthermore, we have $\mathfrak{A}(\mathfrak{H}_n) \leqq \mathfrak{A}(\mathfrak{H}_0)$, where $\mathfrak{H}_n$ and $\mathfrak{H}_0$ are given by (12) and (7) respectively. In other words, *every one of the harmonic surfaces* $\mathfrak{H}_n$ *has an area* $\leqq$ *the area of the initial harmonic surface* $\mathfrak{H}_0$.

2.4. To prove the preceding assertions, let us put

$$\mathfrak{A}_n = \mathfrak{A}(\mathfrak{H}_n) = \iint (E_nG_n - F_n^2)^{1/2}, \quad \mathfrak{J}_n = \frac{1}{2}\iint (E_n + G_n).$$

† If a, b are two real numbers, then max (a, b) denotes the greater one of the two numbers (or their common value, if they are equal).

Then (11) yields the relations

$$\mathfrak{A}_1 \leqq \mathfrak{J}_1 \leqq \mathfrak{A}_0 \leqq \mathfrak{J}_0.$$

Since $\mathfrak{H}_{n+1}$ is derived from $\mathfrak{H}_n$ by the same procedure as $\mathfrak{H}_1$ from $\mathfrak{H}_0$, we have quite generally for $n=0, 1, 2, \cdots$ the relations

$$\mathfrak{A}_{n+1} \leqq \mathfrak{J}_{n+1} \leqq \mathfrak{A}_n \leqq \mathfrak{J}_n. \tag{13}$$

From (13) it follows that the sequences $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \cdots$ and $\mathfrak{J}_0, \mathfrak{J}_1, \mathfrak{J}_2, \cdots$ are both descending. Since all the terms are $\geqq 0$, both sequences are convergent, and from (13) it follows then immediately that *both sequences converge toward the same limit*. Hence if we put $\sigma_n = \mathfrak{J}_n - \mathfrak{A}_n$, then

$$0 \leqq \sigma_n = \mathfrak{J}_n - \mathfrak{A}_n \rightarrow 0.$$

Since $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \cdots$ is a descending sequence, we have also

$$\mathfrak{A}_n \leqq \mathfrak{A}_0. \tag{14}$$

Suppose in the sequel that $n \geqq 1$. From the relations

$$\mathfrak{A}_n = \iint (E_nG_n - F_n^2)^{1/2} \leqq \iint E_n^{1/2}G_n^{1/2} \leqq \frac{1}{2}\iint (E_n + G_n) \tag{15}$$

we infer

$$\begin{aligned} \frac{1}{2}\iint (E_n^{1/2} - G_n^{1/2})^2 &= \iint [\tfrac{1}{2}(E_n + G_n) - E_n^{1/2}G_n^{1/2}] \\ &= \mathfrak{J}_n - \iint E_n^{1/2}G_n^{1/2} \leqq \mathfrak{J}_n - \mathfrak{A}_n = \sigma_n. \end{aligned} \tag{16}$$

Furthermore, since

$$|F_n| = [E_n^{1/2}G_n^{1/2} + (E_nG_n - F_n^2)^{1/2}]^{1/2}[E_n^{1/2}G_n^{1/2} - (E_nG_n - F_n^2)^{1/2}]^{1/2},$$

we infer, from (15), (16), and from the inequality of Schwarz, that

$$\begin{aligned} \left(\iint |F_n|\right)^2 &\leqq \iint [E_n^{1/2}G_n^{1/2} + (E_nG_n - F_n^2)^{1/2}] \times (\mathfrak{J}_n - \mathfrak{A}_n) \\ &\leqq (\mathfrak{J}_n + \mathfrak{A}_n)(\mathfrak{J}_n - \mathfrak{A}_n) = \sigma_n(\mathfrak{J}_n + \mathfrak{A}_n). \end{aligned} \tag{17}$$

On account of (13) we have, since $n \geqq 1$, the inequalities $\mathfrak{A}_n \leqq \mathfrak{J}_n \leqq \mathfrak{A}_0$. Hence, from (17),

$$\iint |F_n| \leqq 2^{1/2}\sigma_n^{1/2}\mathfrak{A}_0^{1/2}. \tag{18}$$

From (16) and (18) it follows that

$$\epsilon_n = \max\left(\iint (E_n^{1/2} - G_n^{1/2})^2, \iint |F_n|\right) \leqq \max(2\sigma_n, 2^{1/2}\sigma_n^{1/2}\mathfrak{A}_0^{1/2}).$$

Since $\mathfrak{A}_0$ is finite and $\sigma_n \to 0$, this proves that $\epsilon_n \to 0$. The last part of the statement in §2.3 has already been proved by (14).

2.5. We apply the preceding results first to the following situation. Let there be given, in the xyz-space, a simple closed polygon $\mathfrak{p}^*$, and let there also be given a polyhedron $\mathfrak{P}$ bounded by $\mathfrak{p}^*$. Given further three distinct points A, B, C on $u^2+v^2=1$, three distinct points A^*, B^*, C^* on $\mathfrak{p}^*$, and given an $\epsilon > 0$.

Then there exists a solution

$$S: x = x(u, v), y = y(u, v), z = z(u, v), u^2 + v^2 \leqq 1,$$

of the problem $P(\mathfrak{p}^*; A, B, C, A^*, B^*, C^*; \epsilon)$ *such that* $\mathfrak{A}(S) \leqq \mathfrak{A}(\mathfrak{P})$. In particular (disregarding the condition concerned with the areas), ***the problem*** $P(\mathfrak{p}^*; A, B, C, A^*, B^*, C^*; \epsilon)$ *has a solution for every choice of* $A, B, C, A^*, B^*, C^*, \epsilon$.

To see this, let

$$\mathfrak{P}: x = \tilde{x}(u, v), y = \tilde{y}(u, v), z = \tilde{z}(u, v), u^2 + v^2 \leqq 1,$$

be an isothermic representation of $\mathfrak{P}$, such that A, B, C are taken into A^*, B^*, C^*. Let $x_0(u, v), y_0(u, v), z_0(u, v)$ be the harmonic functions coinciding with $\tilde{x}(u, v), \tilde{y}(u, v), \tilde{z}(u, v)$ on $u^2+v^2=1$, and denote by $\mathfrak{H}_0$ the surface

$$\mathfrak{H}_0: x = x_0(u, v), y = y_0(u, v), z = z_0(u, v), u^2 + v^2 \leqq 1.$$

On account of the minimizing property of harmonic functions we have again

$$\iint (E_0 + G_0) \leqq \iint (\tilde{E} + G). \tag{19}$$

From the inequalities $(E_0G_0 - F_0^2)^{1/2} \leqq \iint E_0^{1/2}G_0^{1/2} \leqq \iint (E_0 + G_0)/2$, from the equations $\tilde{E} = \tilde{G}$, $\tilde{F} = 0$ and from (19) we infer that

$$\mathfrak{A}(\mathfrak{H}_0) \leqq \frac{1}{2}\iint (E_0 + G_0) \leqq \frac{1}{2}\iint (\tilde{E} + \tilde{G}) = \mathfrak{A}(\mathfrak{P}). \tag{20}$$

Thus the area of $\mathfrak{H}_0$ is finite. Hence we can start up the iterative process, beginning with $\mathfrak{H}_0$. There results a sequence of solutions

$$\mathfrak{H}_n: x = x_n(u, v), y = y_n(u, v), z = z_n(u, v), u^2 + v^2 \leqq 1,$$

of the problems $P(\mathfrak{p}^*; A, B, C, A^*, B^*, C^*; \epsilon_n)$, where $\epsilon_n \to 0$ and $\mathfrak{A}(\mathfrak{H}_n) \leqq \mathfrak{A}(\mathfrak{H}_0)$. Hence, on account of (20), $\mathfrak{A}(\mathfrak{H}_n) \leqq \mathfrak{A}(\mathfrak{P})$ for every n, while ϵ_n will be $< \epsilon$ for n large enough. Thus, for n large enough, $\mathfrak{H}_n$ can be used as the surface S whose existence has been asserted at the beginning of the present section.

2.6. Consider next a Jordan curve Γ^* in the xyz-space. Given three distinct points A, B, C on $u^2+v^2=1$, three distinct points A^*, B^*, C^* on Γ^*, and an $\epsilon>0$. *Then there exists a solution of the problem* $P(\Gamma^*; A, B, C, A^*, B^*, C^*; \epsilon)$.

To see this, observe first that we can approximate Γ^*, in the sense of Fréchet, by simple closed polygons. Hence we have a polygon $\mathfrak{p}^*$ such that $d(\mathfrak{p}^*, \Gamma^*) < \epsilon/2$. From the definition of the distance of two curves it follows then that we have on $\mathfrak{p}^*$ three distinct points $\bar{A}^*$, $\bar{B}^*$, $\bar{C}^*$ such that the distances $A^*\bar{A}^*$, $B^*\bar{B}^*$, $C^*\bar{C}^*$ are all three $<\epsilon/2$. On account of §2.5, the problem $P(\mathfrak{p}^*; A, B, C, \bar{A}^*, \bar{B}^*, \bar{C}^*; \epsilon/2)$ has a solution, and this solution is clearly a solution of the problem $P(\Gamma^*; A, B, C, A^*, B^*, C^*; \epsilon)$, as follows immediately from the definition of this problem (see §1.19)

2.7. Consider finally a Jordan curve Γ^* which bounds some continuous surface, of the topological type of the circular disc, with a finite area. Then the greatest lower bound $\mathfrak{a}(\Gamma^*)$ of the areas of all such surfaces (bounded by Γ^*) is finite. Given three distinct points A, B, C on $u^2+v^2=1$, three distinct points A^*, B^*, C^* on Γ^*, and an $\epsilon>0$. *Then there exists a solution*

$$S: x = x(u, v),\ y = y(u, v),\ z = z(u, v),\ u^2 + v^2 \leqq 1,$$

of the problem $P(\Gamma^*; A, B, C, A^*, B^*, C^*; \epsilon)$, *such that* $\mathfrak{A}(S) \leqq \mathfrak{a}(\Gamma^*) + \epsilon$.

To see this, observe first that we have, on account of the definition of $\mathfrak{a}(\Gamma^*)$, a continuous surface S_0, bounded by Γ^*, such that

$$\mathfrak{A}(S_0) < \mathfrak{a}(\Gamma^*) + \frac{\epsilon}{2}\cdot \tag{21}$$

On account of the definition of the area, we have then a polyhedron $\mathfrak{P}$, such that

$$d(S_0, \mathfrak{P}) < \frac{\epsilon}{2}, \quad \mathfrak{A}(\mathfrak{P}) < \mathfrak{A}(S_0) + \frac{\epsilon}{2}\cdot \tag{22}$$

On account of the definition of $d(S_0, \mathfrak{P})$, the distance of the boundary polygon $\mathfrak{p}^*$ of $\mathfrak{P}$ and of Γ^* is $<\epsilon/2$. Hence we have three distinct points $\bar{A}^*$, $\bar{B}^*$, $\bar{C}^*$ on $\mathfrak{p}^*$ such that the distances $A^*\bar{A}^*$, $B^*\bar{B}^*$, $C^*\bar{C}^*$ are all three $<\epsilon/2$. On account of §2.5, we have then a solution S of the problem $P(\mathfrak{p}^*; A, B, C, A^*, B^*, C^*; \epsilon/2)$ such that $\mathfrak{A}(S) \leqq \mathfrak{A}(\mathfrak{P})$. This S is then clearly a solution of the problem $P(\Gamma^*; A, B, C, A^*, B^*, C^*; \epsilon)$ and it also satisfies the inequality $\mathfrak{A}(S) < \mathfrak{a}(\Gamma^*) + \epsilon$, on account of (21) and (22).

3. A SELECTION THEOREM

3.1. Let Γ^* be a Jordan curve in the xyz-space. Choose three distinct points A, B, C on $u^2+v^2=1$ and three distinct points A^*, B^*, C^* on Γ^*. Suppose we have a sequence $\epsilon_n>0$, $\epsilon_n\to 0$, such that the problem $P(\Gamma^*; A, B, C, A^*, B^*, C^*; \epsilon_n)$ has a solution

$$(23)\qquad S_n\colon x = x_n(u, v),\ y = y_n(u, v),\ z = z_n(u, v),\ u^2 + v^2 \leqq 1,$$

for $n=1, 2, 3, \cdots$.

Then the sequence (23) *contains a subsequence* S_{n_k} *such that* $x_{n_k}(u, v)$, $y_{n_k}(u, v)$, $z_{n_k}(u, v)$ *converge uniformly in* $u^2+v^2\leqq 1$. *The limit surface*

$$S\colon x = x(u, v),\ y = y(u, v),\ z = z(u, v),\ u^2 + v^2 \leqq 1,$$

is a solution of the problem $P(\Gamma^*; A, B, C, A^*, B^*, C^*; 0)$. *If* $\liminf \mathfrak{A}(S_{n_k})$ *is finite, then* $\mathfrak{A}(S)$ *is also finite and satisfies the inequality*

$$\mathfrak{A}(S) \leqq \liminf \mathfrak{A}(S_{n_k}).$$

3.2. This theorem includes, as special cases, a number of selection theorems used previously in the literature.† The proof of the theorem runs as follows. If we put, for the sake of clarity,

$$\xi_n(\theta) = x_n(\cos\theta, \sin\theta),\ \eta_n(\theta) = y_n(\cos\theta, \sin\theta),\ \zeta_n(\theta) = z_n(\cos\theta, \sin\theta),$$

then the equations

$$x = \xi_n(\theta),\ y = \eta_n(\theta),\ z = \zeta_n(\theta)$$

define by assumption a sequence of monotonic transformations, and we can apply to this sequence the generalization of the selection theorem of Helly (see §1.3). There exists therefore a subsequence

$$x = \xi_{n_k}(\theta),\ y = \eta_{n_k}(\theta),\ z = \zeta_{n_k}(\theta)$$

which converges everywhere on the unit circle $u=\cos\theta$, $v=\sin\theta$. The limit functions $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ define a monotonic transformation

$$(24)\qquad T\colon x = \xi(\theta),\ y = \eta(\theta),\ z = \zeta(\theta)$$

† R. Garnier, *Sur le problème de Plateau*, Annales Scientifiques de l'Ecole Normale, vol. 45 (1928), pp. 53–144; T. Radó, *Some remarks on the problem of Plateau*, Proceedings of the National Academy of Sciences, vol. 16 (1930), pp. 242–248, and Annals of Mathematics and Mathematische Zeitschrift, loc. cit.; J. Douglas, loc. cit. in the first foot note on p. 871.

In the development of my own work, I was guided by the analogy with a theorem, stated by Carathéodory, concerning the conformal maps of variable plane Jordan regions (see R. Courant, *Über eine Eigenschaft der Abbildungsfunktionen bei konformen Abbildung*, Göttinger Nachrichten, 1914 and a notice in 1922; T. Radó, *Sur la représentation conforme de domaines variables*, Acta Szeged, vol. 1 (1923)).

of the unit circle $u=\cos\theta$, $y=\sin\theta$ into a set on Γ^*, such that A, B, C are taken into A^*, B^*, C^*. Denote by $x(u, v)$, $y(u, v)$, $z(u, v)$ the harmonic functions obtained by means of the Poisson integral formula, using $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ as boundary functions.† We are going to discuss the surface

$$S\colon x = x(u, v),\ y = y(u, v),\ z = z(u, v), \tag{25}$$

which is defined, for the time being, only in $u^2+v^2<1$.

3.3. It follows from the Poisson integral formula that the harmonic functions $x_{n_k}(u, v)$, $y_{n_k}(u, v)$, $z_{n_k}(u, v)$ and all of their partial derivatives converge, in $u^2+v^2<1$, toward $x(u, v)$, $y(u, v)$, $z(u, v)$ and their corresponding partial derivatives, the convergence being uniform in every concentric circle $u^2+v^2\leqq r^2<1$. Consequently we have

$$\iint_{(r)} (E_{n_k}^{1/2} - G_{n_k}^{1/2})^2 \to \iint_{(r)} (E^{1/2} - G^{1/2})^2, \tag{26}$$

$$\iint_{(r)} |F_{n_k}| \to \iint_{(r)} |F|, \tag{27}$$

$$\iint_{(r)} (E_{n_k}G_{n_k} - F_{n_k}^2)^{1/2} \to \iint_{(r)} (EG - F^2)^{1/2}, \tag{28}$$

for every r such that $0<r<1$, the symbol (r) indicating that the integrals are taken over $u^2+v^2<r^2$. We also have

$$\iint_{(r)} (E_{n_k}^{1/2} - G_{n_k}^{1/2})^2 \leqq \iint_{(1)} (E_{n_k}^{1/2} - G_{n_k}^{1/2})^2 \leqq \epsilon_{n_k}, \tag{29}$$

$$\iint_{(r)} |F_{n_k}| \leqq \iint_{(1)} |F_{n_k}| \leqq \epsilon_{n_k}, \tag{30}$$

$$\iint_{(r)} (E_{n_k}G_{n_k} - F_{n_k}^2)^{1/2} \leqq \iint_{(1)} (E_{n_k}G_{n_k} - F_{n_k}^2)^{1/2} = \mathfrak{A}(S_{n_k}), \tag{31}$$

where the last line is, of course, to be considered only if $\mathfrak{A}(S_{n_k})$ is finite. From (26) to (31) it follows then, on account of $\epsilon_{n_k}\to 0$, that

$$\iint_{(r)} (E^{1/2} - G^{1/2})^2 = 0, \quad \iint_{(r)} |F| = 0, \tag{32}$$

$$\iint_{(r)} (EG - F^2)^{1/2} \leqq \liminf \mathfrak{A}(S_{n_k}). \tag{33}$$

From (33) it follows, in case $\liminf \mathfrak{A}(S_{n_k})$ happens to be finite, for $r\to 1$ that

† On account of §1.5, (d), the functions $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ are integrable even in the Riemann sense.

$$\mathfrak{A}(S) = \iint_{(1)} (EG - F^2)^{1/2} \leqq \liminf \mathfrak{A}(S_{n_k}). \tag{34}$$

From (32) it follows, since E, F, G are continuous (and even analytic), that $E=G$, $F=0$ for $u^2+v^2<r^2$. Since r is arbitrary, it follows that

$$E = G, F = 0 \text{ in } u^2 + v^2 < 1. \tag{35}$$

3.4. Since (24) is a monotonic transformation, $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ have the properties listed in §1.5. On account of (35), it follows therefore from the lemma of Douglas (see §1.18), in connection with §1.5, (c), that $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ are continuous on the whole unit circle $u=\cos\theta$, $v=\sin\theta$. Consequently the harmonic functions $x(u, v), y(u, v), z(u, v)$ remain continuous on $u^2+v^2=1$ (that is to say, they are continuous in $u^2+v^2\leqq 1$), and we have

$$x(u, v) = \xi(\theta),\ y(u, v) = \eta(\theta),\ z(u, v) = \zeta(\theta) \text{ on } u^2 + v^2 = 1. \tag{36}$$

From this it follows that the equations (24) carry $u^2+v^2=1$ in a one-to-one way into Γ^*. Otherwise there would exist (see §1.5 (a)) an arc σ of $u^2+v^2=1$, such that $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ all three reduce to constants on σ. On account of (36) and (35) it would then follow from §1.16 that $x(u, v)$, $y(u, v)$, $z(u, v)$ and consequently (cf. (36)) also $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ reduce to constants identically. This contradicts the fact that the equations (24) carry the points A, B, C into three distinct points A^*, B^*, C^*.

3.5. Since it has been established in §3.4 that $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$ are continuous, it follows from §1.4 that $\xi_{n_k}(\theta)$, $\eta_{n_k}(\theta)$, $\zeta_{n_k}(\theta)$ converge uniformly toward $\xi(\theta)$, $\eta(\theta)$, $\zeta(\theta)$. From the principle of maximum it follows then that the harmonic functions $x_{n_k}(u, v)$, $y_{n_k}(u, v)$, $z_{n_k}(u, v)$ converge uniformly toward $x(u, v)$, $y(u, v)$, $z(u, v)$ in $u^2+v^2\leqq 1$. This completes the proof of the theorem stated in §3.1.

4. Applications

4.1. Let there be given, in the xyz-space, a Jordan curve Γ^*. Given three distinct points A, B, C on $u^2+v^2=1$, and three distinct points A^*, B^*, C^* on Γ^*. Then the problem $P(\Gamma^*; A, B, C, A^*, B^*, C^*; 1/n)$ is solvable for every positive integer n (see §2.6). Denote by

$$S_n\colon x = x_n(u, v),\ y = y_n(u, v),\ z = z_n(u, v),\ u^2 + v^2 \leqq 1, \tag{37}$$

a solution of this problem. On account of the selection theorem (see §3.1), a properly chosen subsequence of (37) will converge uniformly in $u^2+v^2\leqq 1$, and the limit surface solves the problem $P(\Gamma^*; A, B, C, A^*, B^*, C^*; 0)$, that is to say, the problem of Plateau for Γ^* (see §1.20). In other words: *the problem of Plateau is solvable for every Jordan curve.*†

† This result was first obtained by J. Douglas, loc. cit. in the first foot note on p. 871.

4.2. Suppose now that Γ^* bounds some continuous surface, of the topological type of the circular disc, with a finite area. Then the greatest lower bound $\mathfrak{a}(\Gamma^*)$ of the areas of all continuous surfaces, of the topological type of the circular disc and bounded by Γ^*, is finite. The problem $P(\Gamma^*; A, B, C, A^*, B^*, C^*; 1/n)$ has then (see §2.7) a solution

$$S_n: x = x_n(u, v), \; y = y_n(u, v), \; z = z_n(u, v), \; u^2 + v^2 \leqq 1,$$

such that

$$\mathfrak{A}(S_n) < \mathfrak{a}(\Gamma^*) + 1/n. \tag{38}$$

On account of the selection theorem of §3.1, a properly chosen subsequence S_{n_k} will converge, uniformly in $u^2+v^2 \leqq 1$, toward a solution S of the problem $P(\Gamma^*; A, B, C, A^*, B^*, C^*; 0)$, such that

$$\mathfrak{A}(S) \leqq \liminf \mathfrak{A}(S_{n_k}). \tag{39}$$

From (38) and (39) it follows that $\mathfrak{A}(S) \leqq \mathfrak{a}(\Gamma^*)$. On the other hand we have also $\mathfrak{A}(S) \geqq \mathfrak{a}(\Gamma^*)$, since S is a continuous surface, of the type of the circular disc, bounded by Γ^*. Hence $\mathfrak{A}(S) = \mathfrak{a}(\Gamma^*)$. In other words: *if S bounds some continuous surface (of the topological type of the circular disc) with a finite area, then there exists a minimal surface, bounded by Γ^*, whose area is a minimum with respect to all continuous surfaces, of the topological type of the circular disc, bounded by Γ^*.*†

4.3. Suppose now that Γ^* is a simple closed polygon $\mathfrak{p}^*$. Then, first of all, the construction used in §2.5 shows that we have a harmonic surface

$$\mathfrak{H}_0: x = x_0(u, v), \; y = y_0(u, v), \; z = z_0(u, v), \; u^2 + v^2 \leqq 1,$$

bounded by $\mathfrak{p}^*$ and having a finite area. Starting with *any* such surface $\mathfrak{H}_0$, the iterative process yields a sequence

$$\mathfrak{H}_n: x = x_n(u, v), \; y = y_n(u, v), \; z = z_n(u, v), \; u^2 + v^2 \leqq 1, \tag{40}$$

and $\mathfrak{H}_n$ solves the problem $P(\mathfrak{p}^*; A, B, C, A^*, B^*, C^*; \epsilon_n)$, where $\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots$ are positive numbers such that $\epsilon_n \to 0$. On account of the selection theorem, the sequence (40) contains a uniformly convergent subsequence, and the limit surface solves the problem of Plateau for $\mathfrak{p}^*$. It can easily be

† This result was first obtained by the author. See Radó, *The problem of the least area and the problem of Plateau*, Mathematische Zeitschrift, vol. 32 (1930), pp. 763–796. Subsequent proofs have been given by J. Douglas, *Solution of the problem of Plateau*, these Transactions, vol. 33 (1931), pp. 263–321, and, quite recently, by E. J. McShane, *Parametrizations of saddle surfaces, with application to the problem of Plateau*, in the current volume of these Transactions, pp. 716–733. All the proofs of this result known at the present time depend on the passage from S to $\mathfrak{H}$ (see the Intoduction) first used in the author's paper *On Plateau's problem*, Annals of Mathematics, (2), vol. 31 (1930), pp. 457–469.

shown that if we choose the initial harmonic surface $\mathfrak{H}_0$ in all possible ways, then we obtain in this way *all* the solutions of the problem of Plateau. Indeed let

$$S: x = x(u, v),\ y = y(u, v),\ z = z(u, v),\ u^2 + v^2 \leqq 1,$$

be any solution of the problem as stated in §1.14. Then $x(u, v)$, $y(u, v)$, $z(u, v)$ are harmonic in $u^2+v^2<1$, and thus we can use S as the initial harmonic surface of the iterative process, *provided* the area of S is finite. But this is indeed so, on account of a theorem of Carleman. According to Carleman, every minimal surface, of the topological type of the circular disc, satisfies the isoperimetric inequality

$$\mathfrak{A} \leqq \frac{1}{4\pi} L^2$$

where $\mathfrak{A}$ is the area of the surface and L is the length of its perimeter.† In our case, the perimeter is a polygon, and thus L and consequently $\mathfrak{A}$ is finite. Thus S can be used as the initial harmonic surface $\mathfrak{H}_0$ of the iterative process, and it is then obvious, on account of condition (b) of the problem of Plateau (see §1.14), that all the harmonic surfaces $\mathfrak{H}_n$ coincide with S.

4.4. Thus the fact that the iterative process yields all the solutions of the problem of Plateau appears as trivial. On the other hand, I feel that this fact constitutes the specific advantage of the method. My own previous work, as well as the work of Douglas and that of McShane, yielded a solution with a minimum area, and therefore certainly not the general solution.

The iterative process might be considered therefore as a contribution to *the problem of determining the totality of the solutions of the problem of Plateau.* If the given curve has a simply covered convex curve as its parallel or central projection upon some plane, then the solution of the problem is unique.‡ As far as I know, the exact number of the solutions has not yet been determined in any other case.

† See, for a simplified proof and literature, E. F. Beckenbach, *The area and boundary of minimal surfaces*, Annals of Mathematics, vol. 33 (1932), pp. 658–664. Further developments on the isoperimetric inequality are contained in a joint paper by E. F. Beckenbach and the present author, *Subharmonic functions and surfaces of negative curvature*, these Transactions, vol. 35 (1933), pp. 662–674.

‡ See Radó, Acta Szeged, vol. 6 (1932), pp. 1-20.

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